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# Non-equilibrium dynamics of a simple stochastic model

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**Abstract.** We investigate the low-temperature dynamics of a simple stochastic model, introduced recently in the context of the physics of glasses. The slowest characteristic time at equilibrium diverges exponentially at low temperature. On smaller time scales, the non-equilibrium dynamics of the system exhibits an aging regime. We present an analytical study of the scaling behaviour of the mean energy, of its local correlation and response functions, and of the associated fluctuation–dissipation ratio throughout the regime of low temperature and long times. This analysis includes the aging regime, the convergence to equilibrium, and the crossover behaviour between them.

## 1. Introduction

The understanding of non-equilibrium phenomena owes much to the analysis of simple models. For example, kinetic Ising models play an important role in the study of coarsening phenomena [1–3]. The so-called Backgammon model is another example of a simple model which brings useful insight into the fields of non-equilibrium statistical physics, slow dynamics and aging phenomena [4–6]. It was initially introduced in the context of the physics of glasses, as an illustration of the fact that a model possessing entropic barriers only could reproduce some of the features of more complex models possessing both energy and entropy barriers [7]. For example, this model exhibits non-stationary dynamics, with aging of two-times quantities [7–11] and violation of the fluctuation–dissipation theorem [12].

This work is devoted to the behaviour of the Backgammon model at finite temperature. We perform an exact analysis of the convergence of the mean energy towards equilibrium, and of the relaxation of the fluctuations of the energy in the equilibrium state. We also give an analytical treatment of the non-equilibrium dynamics of the model, which becomes asymptotically exact at long times and low temperature. This study is adapted from the theoretical framework introduced in [11]. It allows us to show that the correlation and response functions of the energy are aging at low temperature, to give their scaling forms, and to compute the associated fluctuation–dissipation ratio. This analysis is made possible by a specific feature of the model, already present at equilibrium, which is the existence of a clear-cut separation of slow modes from fast ones, respectively known as  $\alpha$  and  $\beta$  relaxation in the phenomenology of glassy dynamics [6]. A parallel study of the dynamics of the density fluctuations in this model will be the subject of a separate paper [13].

The model is defined as follows. Consider  $N$  particles, initially distributed amongst  $M$  boxes in a known fashion. The energy of a box is equal to 0 if the box is occupied by one

or more particles, and to  $-1$  if it is empty. The total energy of the system is the sum of the energies of each of the boxes; it is thus equal to minus the number of empty boxes. There is no reference to a spatial structure in the definition of the model, hence it is of a mean-field type. At each time step  $1/N$ , a particle and a box are chosen independently at random. The particle is moved to the box according to the Metropolis rule, i.e. with unit rate if the energy does not increase, otherwise with rate  $e^{-\beta}$ , where  $\beta$  is the inverse temperature. In particular, moves from a box containing one particle to an empty box are always allowed, since they do not change the total energy. This dynamics satisfies detailed balance.

The main features of this model are easy to grasp intuitively, and can be explained in simple physical terms. Consider for simplicity the zero-temperature dynamics [7–11]. The moves which decrease the energy consist in emptying the boxes which contain only one particle, to the benefit of a non-empty box. As time passes, the number of empty boxes is ever increasing, and so does the mean occupancy of non-empty boxes. It is therefore more and more rare to find a box with only one particle. The dynamics is thus slowed down by purely entropic effects, since there are no energy barriers.

Finally we recall that the present model is similar to a coarsening system which never equilibrates [1, 2]. The mean occupancy of the non-empty boxes plays the role of the mean domain size in the coarsening system. At zero temperature it grows approximately as  $\Lambda(t) \sim \ln t$ , while the normalized two-time correlation function of the energy has the scaling form [11]

$$C(t, s) \approx \left(\frac{s}{t}\right)^{1/2} \frac{\ln s}{\ln t}. \quad (1.1)$$

## 2. Physical quantities and their dynamical evolution

Consider a finite system, made of  $M$  boxes containing  $N$  particles. Let  $N_i(t)$  be the occupation number of box number  $i$  at time  $t$ , i.e. the number of particles contained in that box. We have

$$\sum_{i=1}^M N_i(t) = N. \quad (2.1)$$

The model is characterized by the following Hamiltonian  $\mathcal{H}$ , and action  $S$ , at inverse temperature  $\beta$ :

$$S = \beta\mathcal{H} = -\beta \sum_{i=1}^M \delta_{N_i, 0}. \quad (2.2)$$

We consider the system in the thermodynamic limit with a fixed density ( $M, N \rightarrow \infty$ ,  $\rho = N/M$  fixed). In the following we will be interested in quantities which involve the occupation number of one box only. Therefore, taking box number 1 as a generic box, we describe the dynamics of the system in terms of  $N_1(t)$  only. We denote by  $f_k(t)$  the probability that box number 1 contains  $k$  particles at time  $t$ :

$$f_k(t) = \text{Prob}\{N_1(t) = k\}. \quad (2.3)$$

Equivalently,  $f_k(t)$  represents the fraction of boxes containing  $k$  particles at time  $t$ .

Restricting the analysis to the homogeneous non-equilibrium initial condition where there is one particle per box, we have  $N_i(t=0) = 1$  for all  $i$ , hence  $\rho = 1$  and

$$f_k(0) = \delta_{k,1}. \quad (2.4)$$

The occupation probabilities  $f_k(t)$  obey the following dynamical equations, whose derivation is given in appendix A:

$$\begin{aligned}\frac{df_k(t)}{dt} &= \frac{k+1}{\Lambda(t)} f_{k+1}(t) + f_{k-1}(t) - \left(1 + \frac{k}{\Lambda(t)}\right) f_k(t) & (k \geq 2) \\ \frac{df_1(t)}{dt} &= \frac{2}{\Lambda(t)} f_2(t) + \mu(t) f_0(t) - 2f_1(t) \\ \frac{df_0(t)}{dt} &= f_1(t) - \mu(t) f_0(t)\end{aligned}\quad (2.5)$$

where

$$\frac{1}{\Lambda(t)} = 1 + (e^{-\beta} - 1) f_0(t) \quad (2.6a)$$

$$\mu(t) = e^{-\beta} + (1 - e^{-\beta}) f_1(t). \quad (2.6b)$$

We introduce the compact notation

$$\frac{df_k(t)}{dt} = \sum_{\ell \geq 0} \mathcal{M}_{k\ell}[f_0(t), f_1(t)] f_\ell(t) \quad (2.7)$$

where the matrix  $\mathcal{M}[f_0(t), f_1(t)]$  satisfies

$$\sum_{k \geq 0} \mathcal{M}_{k\ell}[f_0(t), f_1(t)] = 0. \quad (2.8)$$

The dynamical equations (2.5) are non-linear, since the transition rates involve  $\Lambda(t)$  and  $\mu(t)$ , and thus depend on  $f_0(t)$  and  $f_1(t)$ . We notice that the equation for  $f_0(t)$  can be recast as  $df_0(t)/dt = f_1(t)/\Lambda(t) - e^{-\beta} f_0(t)$ . Finally, equations (2.5) ensure the conservation of the moments

$$\sum_{k \geq 0} f_k(t) = 1 \quad \langle N_1(t) \rangle = \sum_{k \geq 1} k f_k(t) = \rho = 1. \quad (2.9)$$

The dynamical equations (2.5) have a simple interpretation. If  $N_1(t) = k \geq 2$ , the rate at which a particle enters box number 1 is unity, while the rate *per particle* for leaving that box is  $1/\Lambda(t)$ . Hence the total departure rate is  $k/\Lambda(t)$ . Therefore, if  $k < \Lambda(t)$  the occupation number of the box has a tendency to increase, while if  $k > \Lambda(t)$  the tendency is opposite. The occupation number  $N_1(t)$  can also be viewed as the position at time  $t$  of a random walker on a semi-infinite chain, taking values  $N_1(t) = k = 0, 1, \dots$ , and obeying the master equation (2.5). The walk is biased to the left if  $k > \Lambda(t)$ , to the right if  $k < \Lambda(t)$ , so that the walker is confined around  $k = \Lambda(t)$  by a restoring force. Hence  $f_k(t)$ , the distribution of  $N_1(t)$ , is expected to be peaked around  $k = \Lambda(t)$  for long times, and for  $\Lambda(t)$  large enough, i.e. at low temperature. Different rules hold at  $k = 0$  and 1. In particular  $k = 0$  plays the role of an absorbing barrier [9].

Note that the average position of the walker  $\langle N_1(t) \rangle$  is fixed by equation (2.9), so that the distribution  $f_k(t)$  will in fact be bimodal at low temperature. This property will be illustrated by the case of the equilibrium distribution (see section 5). Finally, there are two time scales in this system. On the shorter time scale,  $\Lambda(t)$  may be considered as constant, so that there is equilibration in a fixed confining potential. On the longer scale, the potential itself evolves slowly, as  $\Lambda(t)$  increases. These time scales will be identified in sections 5 and 6 with those of  $\alpha$  and  $\beta$  relaxation, found in glassy dynamics.

### 2.1. Mean energy

The first quantity to be studied is the energy of the system. According to equation (2.2), the energy of box number  $i$  is  $E_i(t) = -\delta_{N_i(t),0}$ . The mean energy per box of a thermodynamic system thus reads

$$E(t) = \langle E_1(t) \rangle = -f_0(t). \quad (2.10)$$

It can be expressed in terms of  $\Lambda(t)$  as

$$E(t) = \frac{1 - \Lambda(t)}{(1 - e^{-\beta})\Lambda(t)}. \quad (2.11)$$

At low temperature, we have  $E(t) \approx -1 + 1/\Lambda(t)$  up to an exponentially small correction, proportional to  $e^{-\beta}$ . The mean occupancy of the non-empty boxes is equal to  $1/(1 - f_0(t)) \approx \Lambda(t)$ , again up to an exponentially small correction at low temperature, hence the interpretation of  $\Lambda(t)$  as a characteristic domain size, in analogy with coarsening systems.

### 2.2. The energy correlation function

The correlation function  $c(t, s)$  between the values of the energy  $E_1$  of box number 1 at the two times  $s$  (preparation time, or waiting time) and  $t$  (observation time) is defined as

$$c(t, s) = \langle E_1(t)E_1(s) \rangle - \langle E_1(t) \rangle \langle E_1(s) \rangle \quad (2.12)$$

with  $0 \leq s \leq t$ . Note that the present study is restricted to the local correlation function  $c(t, s)$ , disregarding the non-diagonal correlations between two different boxes.

Since we have  $\langle E_1(t)E_1(s) \rangle = \text{Prob}\{N_1(t) = 0, N_1(s) = 0\}$ , we are led to introduce the joint probabilities  $\text{Prob}\{N_1(t) = k, N_1(s) = 0\}$ . Equivalently, we will consider the conditional probabilities  $g_k(t, s)$  that box number 1 contains  $k$  particles at time  $t$ , knowing that it was empty at the earlier time  $s$ :

$$g_k(t, s) = \text{Prob}\{N_1(t) = k | N_1(s) = 0\}. \quad (2.13)$$

These quantities obey the dynamical equations (see appendix A)

$$\frac{\partial g_k(t, s)}{\partial t} = \sum_{\ell \geq 0} \mathcal{M}_{k\ell} [f_0(t), f_1(t)] g_\ell(t, s) \quad (2.14)$$

with the initial condition

$$g_k(s, s) = \delta_{k,0}. \quad (2.15)$$

Equation (2.14) is now linear in the unknowns  $g_k(t, s)$ . It implies

$$\sum_{k \geq 0} g_k(t, s) = 1 \quad (2.16)$$

at all times  $t \geq s$ .

The energy correlation function then reads

$$c(t, s) = f_0(s)(g_0(t, s) - f_0(t)). \quad (2.17)$$

We also introduce the normalized correlation function

$$C(t, s) = \frac{c(t, s)}{c(s, s)} = \frac{g_0(t, s) - f_0(t)}{1 - f_0(s)} \quad (2.18)$$

such that  $C(s, s) = 1$ .

2.3. The energy response function

The local response function  $r(t, s)$  is a measure of the change in the energy of box number 1 at time  $t$ , induced by an infinitesimal change of the local temperature of the same box at the earlier time  $s$ .

More generally, assume that box number 1 is subjected to an arbitrary time-dependent inverse temperature  $\beta_1(t)$ , while the other boxes ( $i = 2, \dots, M$ ) are subjected to a uniform and constant inverse temperature  $\beta$ . To leading order at large  $M$ , the occupation probabilities of the boxes  $i = 2, \dots, M$  are still given by the  $f_k(t)$ , while box number 1 has a different distribution, depending on  $\beta_1(t)$ , denoted by  $f_k^{\beta_1}(t)$ .

The dynamical equations for the  $f_k^{\beta_1}(t)$  are slightly different from equations (2.5) (see appendix A):

$$\begin{aligned} \frac{df_k^{\beta_1}(t)}{dt} &= \frac{k+1}{\Lambda(t)} f_{k+1}^{\beta_1}(t) + f_{k-1}^{\beta_1}(t) - \left(1 + \frac{k}{\Lambda(t)}\right) f_k^{\beta_1}(t) \quad (k \geq 2) \\ \frac{df_1^{\beta_1}(t)}{dt} &= \frac{2}{\Lambda(t)} f_2^{\beta_1}(t) + \mu_+(t) f_0^{\beta_1}(t) - (1 + \mu_-(t)) f_1^{\beta_1}(t) \\ \frac{df_0^{\beta_1}(t)}{dt} &= \mu_-(t) f_1^{\beta_1}(t) - \mu_+(t) f_0^{\beta_1}(t) \end{aligned} \tag{2.19}$$

with the initial value

$$f_k^{\beta_1}(0) = \delta_{k,1}. \tag{2.20}$$

In equations (2.19) we have set

$$\begin{aligned} \mu_+(t) &= (1 - f_1(t))e^{-\beta_1(t)} + f_1(t) \mathcal{W}(\beta_1(t) - \beta) \\ \mu_-(t) &= 1 - f_0(t) + f_0(t) \mathcal{W}(\beta - \beta_1(t)) \end{aligned} \tag{2.21}$$

where

$$\mathcal{W}(\Delta S) = \min(1, e^{-\Delta S}) \tag{2.22}$$

is the Metropolis acceptance rate associated with a change of action  $\Delta S$  (see equation (A.2)).

The response function is defined as

$$r(t, s) = - \left. \frac{\delta \langle E_1(t) \rangle}{\delta \beta_1(s)} \right|_{\beta_1(s)=\beta}. \tag{2.23}$$

Defining more generally

$$h_k(t, s) = \left. \frac{\delta f_k^{\beta_1}(t)}{\delta \beta_1(s)} \right|_{\beta_1(s)=\beta} \tag{2.24}$$

we have

$$r(t, s) = h_0(t, s). \tag{2.25}$$

The dynamical equations for  $h_k(t, s)$  are obtained by taking the functional derivative of equations (2.19) with respect to  $\beta_1$  for  $\beta_1(s) = \beta$ . This yields

$$\frac{\partial h_k(t, s)}{\partial t} = \sum_{\ell \geq 0} \mathcal{M}_{k\ell} [f_0(t), f_1(t)] h_\ell(t, s) \tag{2.26}$$

with the initial condition

$$h_k(s, s) = (\delta_{k,0} - \delta_{k,1}) \mu(s) f_0(s). \tag{2.27}$$

Therefore

$$\sum_{k \geq 0} h_k(t, s) = 0 \quad (2.28)$$

at all times  $t \geq s$ . It is to be noted that the above equations are not affected by the fact that the derivative of the Metropolis acceptance rate (2.22) is not well defined at  $\Delta S = 0$ . Indeed,  $d\mathcal{W}(\Delta S \rightarrow 0^+)/d\Delta S = -1$  and  $d\mathcal{W}(\Delta S \rightarrow 0^-)/d\Delta S = 0$  do not coincide.

We also introduce the normalized response function

$$R(t, s) = \frac{r(t, s)}{r(s, s)} = \frac{h_0(t, s)}{\mu(s)f_0(s)} \quad (2.29)$$

such that  $R(s, s) = 1$ .

#### 2.4. The fluctuation–dissipation ratio

The fluctuation–dissipation ratio  $X(t, s)$  provides a measure of the departure of the system from equilibrium [6, 14, 12]. In the present case it is defined as

$$r(t, s) = X(t, s) \frac{\partial c(t, s)}{\partial s}. \quad (2.30)$$

This definition contains no explicit temperature dependence, in contrast with the fluctuation–dissipation ratio of, for example, density fluctuations.

The energy fluctuation–dissipation ratio can be expressed in terms of quantities introduced above. Indeed  $\partial g_k(t, s)/\partial s$  obeys the same dynamical equations as  $g_k(t, s)$  or  $h_k(t, s)$  for  $t > s$ , since  $s$  only enters these equations as a parameter. Moreover, let us integrate equation (2.14) for the  $g_k(t, s)$  to first order in  $\varepsilon = t - s$ :

$$g_k(t = s + \varepsilon, s) = \delta_{k,0} + (\delta_{k,1} - \delta_{k,0})\mu(s)\varepsilon + \mathcal{O}\varepsilon^2. \quad (2.31)$$

Hence

$$\frac{\partial g_k(s, s)}{\partial s} = (\delta_{k,0} - \delta_{k,1})\mu(s). \quad (2.32)$$

Comparing this last expression with the initial condition (2.27) for the  $h_k(t, s)$  gives

$$h_k(t, s) = f_0(s) \frac{\partial g_k(t, s)}{\partial s}. \quad (2.33)$$

As a consequence,

$$\frac{\partial c(t, s)}{\partial s} = h_0(t, s) + \frac{df_0(s)}{ds}(g_0(t, s) - f_0(t)) \quad (2.34)$$

so that the fluctuation–dissipation ratio  $X(t, s)$  reads

$$X(t, s) = \frac{h_0(t, s)}{h_0(t, s) + \frac{df_0(s)}{ds}(g_0(t, s) - f_0(t))} \quad (2.35)$$

or else

$$X(t, s) = \frac{r(t, s)}{r(t, s) + \frac{1}{E(s)} \frac{dE(s)}{ds} c(t, s)}. \quad (2.36)$$

We can check from this expression that  $X(t, s) = 1$  at equilibrium, recovering thus the fluctuation–dissipation theorem

$$r(t, s) = \frac{\partial c(t, s)}{\partial s} \quad (2.37)$$

while  $0 < X(t, s) < 1$  out of equilibrium.

### 3. Generating functions and integral representations

It is possible to obtain integral representations of the solutions of equations (2.5), (2.14) and (2.26) by means of the generating functions

$$F(x, t) = \sum_{k \geq 0} f_k(t) x^k \quad G(x, t, s) = \sum_{k \geq 0} g_k(t, s) x^k \quad H(x, t, s) = \sum_{k \geq 0} h_k(t, s) x^k. \tag{3.1}$$

These functions obey the following partial differential equations

$$\frac{\partial}{\partial t} F(x, t) = (x - 1) \left( F(x, t) - \frac{1}{\Lambda(t)} \frac{\partial}{\partial x} F(x, t) - Y_f(t) \right) \tag{3.2a}$$

$$\frac{\partial}{\partial t} G(x, t, s) = (x - 1) \left( G(x, t, s) - \frac{1}{\Lambda(t)} \frac{\partial}{\partial x} G(x, t, s) - Y_g(t, s) \right) \tag{3.2b}$$

$$\frac{\partial}{\partial t} H(x, t, s) = (x - 1) \left( H(x, t, s) - \frac{1}{\Lambda(t)} \frac{\partial}{\partial x} H(x, t, s) - Y_h(t, s) \right) \tag{3.2c}$$

where

$$Y_f(t) = (1 - e^{-\beta}) f_0(t) = 1 - 1/\Lambda(t) \tag{3.3a}$$

$$\begin{aligned} Y_g(t, s) &= (1 - \mu(t)) g_0(t, s) + (1 - 1/\Lambda(t)) g_1(t, s) \\ &= (1 - e^{-\beta}) ((1 - f_1(t)) g_0(t, s) + f_0(t) g_1(t, s)) \\ &= (1 - e^{-\beta}) \left( g_0(t, s) + f_0(t) \frac{\partial g_0(t, s)}{\partial t} - \frac{d f_0(t)}{dt} g_0(t, s) \right). \end{aligned} \tag{3.3b}$$

Similar expressions for  $Y_h(t, s)$  are obtained by replacing  $g$  by  $h$  in the above three expressions for  $Y_g$ . The initial conditions for equations (3.2) are derived from equations (2.4), (2.15) and (2.27):

$$F(x, 0) = x \quad G(x, s, s) = 1 \quad H(x, s, s) = (1 - x) \mu(s) f_0(s) \tag{3.4}$$

while the conservation of moments expressed by equations (2.9), (2.16) and (2.28) implies

$$F(1, t) = \frac{\partial}{\partial x} F(1, t) = 1 \quad G(1, t, s) = 1 \quad H(1, t, s) = 0. \tag{3.5}$$

Equations (3.2) can be formally solved by the method of characteristics (see appendix B), yielding

$$F(x, t) = (1 + (x - 1)e^{-\tau(t)}) e^{(x-1)D(t,0)} + \int_0^t du \mathcal{K}(x, t, u) Y_f(u) \tag{3.6a}$$

$$G(x, t, s) = e^{(x-1)D(t,s)} + \int_s^t du \mathcal{K}(x, t, u) Y_g(u, s) \tag{3.6b}$$

$$H(x, t, s) = \mu(s) f_0(s) \mathcal{K}(x, t, s) + \int_s^t du \mathcal{K}(x, t, u) Y_h(u, s) \tag{3.6c}$$

with the definitions (B.4) and (B.7):

$$\begin{aligned} \tau(t) &= \int_0^t \frac{du}{\Lambda(u)} \\ D(t, u) &= \int_u^t dv e^{\tau(v)-\tau(t)} \\ \mathcal{K}(x, t, u) &= (1 - x) e^{\tau(u)-\tau(t)+(x-1)D(t,u)}. \end{aligned} \tag{3.7}$$



Note that, in agreement with equation (2.33), we have

$$H(x, t, s) = f_0(s) \frac{\partial G(x, t, s)}{\partial s}. \quad (3.8)$$

Setting  $x = 0$  in equations (3.6), we obtain

$$f_0(t) = (1 - e^{-\tau(t)})e^{-D(t,0)} + \int_0^t du \mathcal{K}(0, t, u) Y_f(u) \quad (3.9a)$$

$$g_0(t, s) = e^{-D(t,s)} + \int_s^t du \mathcal{K}(0, t, u) Y_g(u, s) \quad (3.9b)$$

$$h_0(t, s) = \mu(s) f_0(s) \mathcal{K}(0, t, s) + \int_s^t du \mathcal{K}(0, t, u) Y_h(u, s). \quad (3.9c)$$

Equation (3.9a) is an implicit non-linear integral equation for  $\Lambda(t)$ , since  $f_0(t)$ ,  $\tau(t)$ ,  $D(t, u)$ , and  $\mathcal{K}(0, t, u)$  are defined in terms of  $\Lambda(t)$  by equations (2.6a) and (3.7). Let us finally point out the central role played by the kernel

$$\mathcal{K}(0, t, u) = e^{\tau(u) - \tau(t) - D(t,u)} = \frac{\partial}{\partial u} e^{-D(t,u)} \quad (3.10)$$

in the following sections.

#### 4. Infinite-temperature behaviour

At infinite temperature, the energy of configurations plays no role in the dynamics. The rate at which a particle enters any box is constant and equal to unity, while the rate at which a particle leaves the box is equal to unity, per particle. Hence  $\Lambda(t) = \mu(t) = 1$ , and equations (3.6) provide explicit solutions.

##### 4.1. Mean energy

Since  $Y_f(t) = 0$ , equation (3.6a) yields

$$F(x, t) = (1 + (x - 1)e^{-t}) \exp((x - 1)(1 - e^{-t})). \quad (4.1)$$

The occupation probabilities  $f_k(t)$  thus read

$$f_k(t) = ((1 - e^{-t})^2 + ke^{-t}) \frac{(1 - e^{-t})^{k-1} \exp(e^{-t} - 1)}{k!}. \quad (4.2)$$

In particular, the mean energy

$$E(t) = -f_0(t) = (e^{-t} - 1) \exp(e^{-t} - 1) = -\frac{1}{e} + \frac{1}{2e} e^{-2t} + \dots \quad (4.3)$$

converges to its equilibrium value  $E_{\text{eq}} = -e^{-1}$  with a relaxation time  $t_{\text{eq}}^{(1)} = \frac{1}{2}$ .

##### 4.2. Energy correlation and response functions

Similarly, noticing that  $Y_g(t, s) = Y_h(t, s) = 0$ , we obtain

$$\begin{aligned} G(x, t, s) &= \exp((x - 1)(1 - e^{s-t})) \\ H(x, t, s) &= (1 - x)(1 - e^{-s}) \exp(e^{-s} - 1 + s - t + (x - 1)(1 - e^{s-t})). \end{aligned} \quad (4.4)$$

We thus have

$$g_k(t, s) = \frac{(1 - e^{s-t})^k \exp(e^{s-t} - 1)}{k!}$$

$$h_k(t, s) = (1 - e^{-s}) \exp(e^{-s} + e^{s-t} - 2 + s - t) \frac{(1 - e^{s-t})^{k-1} (1 - k - e^{s-t})}{k!}. \tag{4.5}$$

Note that the distributions (4.2), (4.5) thus found are all related to Poisson distributions. In particular the  $g_k(t, s)$  follow an exact Poisson law, with parameter  $1 - e^{s-t}$ .

The energy correlation and response functions then read

$$c(t, s) = (1 - e^{-s}) \exp(e^{-s} - 2) (\exp(e^{s-t}) - (1 - e^{-t}) \exp(e^{-t}))$$

$$r(t, s) = (1 - e^{-s}) \exp(e^{-s} + e^{s-t} - 2 + s - t), \tag{4.6}$$

so that the fluctuation–dissipation ratio is given by

$$X(t, s) = \frac{(1 - e^{-s}) e^{s-t}}{(1 - e^{-s}) e^{s-t} + e^{-2s} (1 - (1 - e^{-t}) \exp(e^{-t} - e^{s-t}))}. \tag{4.7}$$

The equilibrium correlation and response functions only depend on the time difference  $\theta = t - s$ :

$$c_{\text{eq}}(\theta) = e^{-2} (\exp(e^{-\theta}) - 1) \quad r_{\text{eq}}(\theta) = \exp(e^{-\theta} - \theta - 2) \tag{4.8}$$

and they obey the fluctuation–dissipation theorem

$$r_{\text{eq}}(\theta) = - \frac{dc_{\text{eq}}(\theta)}{d\theta}. \tag{4.9}$$

The convergence of  $c(t, s)$  and  $r(t, s)$  to their equilibrium values  $c_{\text{eq}}(\theta)$  and  $r_{\text{eq}}(\theta)$  is in  $e^{-2s}$ , exhibiting again the relaxation time  $t_{\text{eq}}^{(1)} = \frac{1}{2}$ , while the latter quantities fall off as  $e^{-\theta}$ , with a relaxation time  $t_{\text{eq}}^{(2)} = 1$ . The occurrence of these two different relaxation times will be explained in section 5.

### 5. Equilibrium properties and convergence to equilibrium

At finite temperature and for long enough times, the model reaches a stationary state, corresponding to thermal equilibrium. In this section, we analyse the equilibrium behaviour of the quantities defined above (mean energy, correlation and response functions). In particular we compute the relaxation times  $t_{\text{eq}}^{(1)}$  and  $t_{\text{eq}}^{(2)}$  which characterize the convergence of the mean energy to equilibrium, and the relaxation of the local fluctuations of energy at thermal equilibrium, respectively.

#### 5.1. Mean energy

The stationary solution of the dynamical equations (2.5) reads

$$(f_k)_{\text{eq}} = e^{-\Lambda_{\text{eq}}} \frac{\Lambda_{\text{eq}}^{k-1}}{k!} \quad (k \geq 1)$$

$$(f_0)_{\text{eq}} = \frac{\Lambda_{\text{eq}} - 1 + e^{-\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}} = \frac{e^{\beta - \Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}} \tag{5.1}$$

where the value  $\Lambda_{\text{eq}}$  of  $\Lambda(t)$  at equilibrium is related to temperature by

$$e^{\beta} = 1 + (\Lambda_{\text{eq}} - 1) e^{\Lambda_{\text{eq}}}. \tag{5.2}$$

The equilibrium thermodynamics of the model, summarized in appendix C, shows that  $\Lambda_{\text{eq}}$  is to be identified with the fugacity.

The occupation probabilities at equilibrium  $(f_k)_{\text{eq}}$  for  $k \geq 1$  are thus Poissonian with parameter  $\Lambda_{\text{eq}}$ , with the exception of  $(f_0)_{\text{eq}}$ , reflecting that empty boxes are energetically favoured. At low temperature, we have  $\Lambda_{\text{eq}} \approx \beta \gg 1$ , so that the  $(f_k)_{\text{eq}}$  form a strongly bimodal distribution, with many empty boxes, represented by the weight  $(f_0)_{\text{eq}} \approx 1 - 1/\Lambda_{\text{eq}}$  at  $k = 0$ , and few occupied boxes, represented by the weight  $1/\Lambda_{\text{eq}}$  concentrated in a narrow region of width  $\sqrt{\Lambda_{\text{eq}}}$  around  $k = \Lambda_{\text{eq}}$ .

### 5.2. The relaxation time of the mean energy

The relaxation time  $t_{\text{eq}}^{(1)}$  of the mean energy can be derived as follows. First, using equation (3.10), we recast equation (3.9a) into the form

$$f_0(t) = 1 - \mathcal{K}(0, t, 0) - \int_0^t du \frac{\mathcal{K}(0, t, u)}{\Lambda(u)}. \quad (5.3)$$

Next, in order to linearize this expression around equilibrium, we set

$$\Lambda(t) = \Lambda_{\text{eq}} + \delta\Lambda(t). \quad (5.4)$$

We have then, to first order in  $\delta\Lambda(t)$ , setting  $\varepsilon = t - u$ ,

$$f_0(t) = (f_0)_{\text{eq}} + \delta f_0(t) \quad \tau(t) = \frac{t}{\Lambda_{\text{eq}}} + \delta\tau(t) \quad (5.5)$$

$$D(t, u) = \Lambda_{\text{eq}}(1 - e^{-\varepsilon/\Lambda_{\text{eq}}}) + \delta D(t, u) \quad \mathcal{K}(0, t, u) = K(\varepsilon, \Lambda_{\text{eq}}) + \delta\mathcal{K}(0, t, u)$$

where

$$\begin{aligned} \delta f_0(t) &\approx \frac{\Lambda_{\text{eq}} - 1 + e^{-\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}^2 (\Lambda_{\text{eq}} - 1)} \delta\Lambda(t) \\ \delta D(t, u) &\approx \frac{1}{\Lambda_{\text{eq}}} \int_0^\varepsilon d\zeta (e^{-\zeta/\Lambda_{\text{eq}}} - e^{-\varepsilon/\Lambda_{\text{eq}}}) \delta\Lambda(t - \zeta) \end{aligned} \quad (5.6)$$

and where the equilibrium kernel  $K(\varepsilon, \Lambda_{\text{eq}}) = \lim_{t \rightarrow \infty} \mathcal{K}(0, t, t - \varepsilon)$  reads

$$K(\varepsilon, \Lambda_{\text{eq}}) = \exp(-\varepsilon/\Lambda_{\text{eq}} - \Lambda_{\text{eq}}(1 - e^{-\varepsilon/\Lambda_{\text{eq}}})) \quad (5.7)$$

This expression has the following simple relationship to the occupation probabilities  $(f_k)_{\text{eq}}$ :

$$K(\varepsilon, \Lambda_{\text{eq}}) = \sum_{k \geq 1} k (f_k)_{\text{eq}} e^{-k\varepsilon/\Lambda_{\text{eq}}}. \quad (5.8)$$

For long enough times, the linearized form of equation (5.3) reads

$$\delta f_0(t) \approx \frac{1}{\Lambda_{\text{eq}}^2} \int_0^t du K(t - u, \Lambda_{\text{eq}}) \delta\Lambda(u) - \frac{1}{\Lambda_{\text{eq}}} \int_0^t du \delta\mathcal{K}(0, t, u). \quad (5.9)$$

Using equations (3.10) and (5.6), we obtain

$$\delta f_0(t) \approx \frac{1}{\Lambda_{\text{eq}}^2} \int_0^\infty d\varepsilon (K(\varepsilon, \Lambda_{\text{eq}}) - e^{-\Lambda_{\text{eq}} - \varepsilon/\Lambda_{\text{eq}}}) \delta\Lambda(t - \varepsilon). \quad (5.10)$$

Taking the Laplace transform of equation (5.10) leads to the characteristic equation

$$\frac{\Lambda_{\text{eq}} - 1 + e^{-\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}} - 1} = \hat{K}(p, \Lambda_{\text{eq}}) - \frac{e^{-\Lambda_{\text{eq}}}}{p + 1/\Lambda_{\text{eq}}} \quad (5.11)$$

where the Laplace transform  $\hat{K}(p, \Lambda_{\text{eq}})$  of  $K(\varepsilon, \Lambda_{\text{eq}})$  reads

$$\begin{aligned} \hat{K}(p, \Lambda_{\text{eq}}) &= \int_0^\infty d\varepsilon e^{-p\varepsilon} K(\varepsilon, \Lambda_{\text{eq}}) = \Lambda_{\text{eq}} \int_0^1 dz z^{p\Lambda_{\text{eq}}} e^{\Lambda_{\text{eq}}(z-1)} \\ &= e^{-\Lambda_{\text{eq}}} \sum_{k \geq 1} \frac{1}{p + k/\Lambda_{\text{eq}}} \frac{\Lambda_{\text{eq}}^{k-1}}{(k-1)!}. \end{aligned} \tag{5.12}$$

Notice that the subtracted term in the right-hand side of equation (5.11) is just the first term ( $k = 1$ ) in the last expansion of  $\hat{K}(p, \Lambda_{\text{eq}})$ .

Equation (5.11) has an infinite sequence of real negative solutions, which we denote by  $p = -p_k^{(1)}$ , with  $k \geq 2$ . The characteristic values  $p_k^{(1)}$  represent the inverse characteristic times of the relaxation of energy to its equilibrium value. In particular, the relaxation time  $t_{\text{eq}}^{(1)}$  is the inverse of the smallest characteristic value:

$$t_{\text{eq}}^{(1)} = \frac{1}{p_2^{(1)}}. \tag{5.13}$$

At infinite temperature ( $\Lambda_{\text{eq}} = 1$ ), the characteristic values are the poles of  $\hat{K}(p, \Lambda_{\text{eq}})$  (except the first one, because of the subtraction), hence  $p_k^{(1)} = k$  with  $k \geq 2$ , and  $t_{\text{eq}}^{(1)} = \frac{1}{2}$ . As temperature decreases, the spectrum of characteristic values is continuously deformed, and the solutions of equation (5.11) stay within a bounded distance from the poles of  $\hat{K}(p, \Lambda_{\text{eq}})$ , namely  $p_k^{(1)} \approx -k/\Lambda_{\text{eq}}$ . At low temperature, the relaxation time  $t_{\text{eq}}^{(1)}$  is exponentially large in  $\Lambda_{\text{eq}}$ . Indeed, we have the expansion around  $p = 0$

$$\hat{K}(p, \Lambda_{\text{eq}}) = 1 - e^{-\Lambda_{\text{eq}}} - p\Lambda_{\text{eq}}e^{-\Lambda_{\text{eq}}}I(\Lambda_{\text{eq}}) + \dots \tag{5.14}$$

where [11]

$$I(\Lambda) = \int_0^1 \frac{dz}{z} (e^{\Lambda z} - 1) = \sum_{n \geq 1} \frac{\Lambda^n}{n n!} \approx \frac{e^\Lambda}{\Lambda} \sum_{\ell \geq 0} \frac{\ell!}{\Lambda^\ell}. \tag{5.15}$$

By inserting the expansion (5.14) into equation (5.11), we obtain

$$t_{\text{eq}}^{(1)} \approx \frac{\Lambda_{\text{eq}} - 1}{\Lambda_{\text{eq}}} I(\Lambda_{\text{eq}}) \tag{5.16}$$

i.e.

$$t_{\text{eq}}^{(1)} \approx \frac{e^{\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}} \left( 1 + \frac{1}{\Lambda_{\text{eq}}^2} + \dots \right) \approx \frac{e^\beta}{\beta^2} \left( 1 + \frac{2 \ln \beta + 1}{\beta} + \dots \right) \tag{5.17}$$

while all the other characteristic times are of order  $\Lambda_{\text{eq}} \approx \beta$  at low temperature.

### 5.3. The energy correlation and response functions: generalities

At equilibrium, namely for  $s \gg t_{\text{eq}}^{(1)}$ , the energy correlation and response functions become stationary, i.e. invariant under time translation. They only depend on the time difference  $\theta = t - s$ :

$$c_{\text{eq}}(\theta) = (f_0)_{\text{eq}}((g_0)_{\text{eq}}(\theta) - (f_0)_{\text{eq}}) \tag{5.18a}$$

$$r_{\text{eq}}(\theta) = (h_0)_{\text{eq}}(\theta). \tag{5.18b}$$

Their expressions can be derived from equations (3.9b) and (3.9c), which respectively become, setting  $\varepsilon = t - u$  in the integrals,

$$(g_0)_{\text{eq}}(\theta) = e^{-\Lambda_{\text{eq}}(1 - e^{-\theta/\Lambda_{\text{eq}}})} + \int_0^\theta d\varepsilon K(\varepsilon, \Lambda_{\text{eq}})(Y_g)_{\text{eq}}(\theta - \varepsilon) \quad (5.19a)$$

$$(h_0)_{\text{eq}}(\theta) = e^{-\Lambda_{\text{eq}}} K(\theta, \Lambda_{\text{eq}}) + \int_0^\theta d\varepsilon K(\varepsilon, \Lambda_{\text{eq}})(Y_h)_{\text{eq}}(\theta - \varepsilon) \quad (5.19b)$$

where the equilibrium kernel  $K(\varepsilon, \Lambda_{\text{eq}})$  has been defined in equation (5.7), and where

$$(Y_g)_{\text{eq}}(\theta) = (1 - e^{-\beta})((1 - (f_1)_{\text{eq}})(g_0)_{\text{eq}}(\theta) + (f_0)_{\text{eq}}(g_1)_{\text{eq}}(\theta)) \quad (5.20)$$

and a similar expression for  $(Y_h)_{\text{eq}}$ . First of all, we have the identity

$$(h_0)_{\text{eq}}(\theta) = -(f_0)_{\text{eq}} \frac{d(g_0)_{\text{eq}}(\theta)}{d\theta} \quad (5.21)$$

(see equation (2.33)), which expresses the fluctuation–dissipation theorem (2.37) in the form

$$r_{\text{eq}}(\theta) = -\frac{dc_{\text{eq}}(\theta)}{d\theta} \quad (5.22)$$

generalizing equation (4.9) to any finite temperature.

The integral equations (5.19) can be solved by Laplace transforms. Indeed, using the definition of  $Y_h$  and the dynamical equation (2.26) for  $h_0$ , we obtain

$$(\hat{h}_0)_{\text{eq}}(p) = (f_0)_{\text{eq}}(1 - p(\hat{g}_0)_{\text{eq}}(p)) = \frac{e^{-\Lambda_{\text{eq}}} \hat{K}(p, \Lambda_{\text{eq}})}{\Lambda_{\text{eq}} - (\Lambda_{\text{eq}} - 1) \left(p + \frac{1}{(f_0)_{\text{eq}}}\right) \hat{K}(p, \Lambda_{\text{eq}})}. \quad (5.23)$$

The function in the rightmost side of equation (5.23) is a meromorphic function, with an infinite sequence of poles, situated on the real negative axis, which we denote by  $p = -p_k^{(2)}$ , with  $k \geq 1$ . The  $p_k^{(2)}$  give the inverse characteristic times of the energy correlation and response functions. In particular the relaxation time of these functions is the inverse of the smallest pole:

$$t_{\text{eq}}^{(2)} = \frac{1}{p_1^{(2)}}. \quad (5.24)$$

We have therefore

$$r_{\text{eq}}(\theta) = (h_0)_{\text{eq}}(\theta) = \sum_{k \geq 1} a_k e^{-p_k^{(2)} \theta} \quad (5.25a)$$

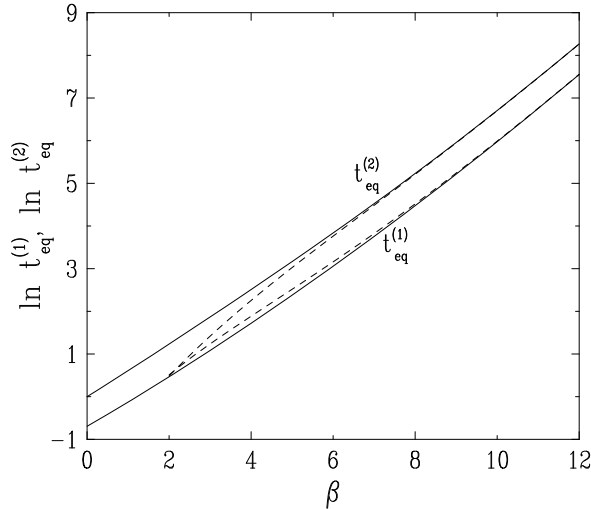
$$c_{\text{eq}}(\theta) = (f_0)_{\text{eq}}((g_0)_{\text{eq}}(\theta) - (f_0)_{\text{eq}}) = \sum_{k \geq 1} \frac{a_k}{p_k^{(2)}} e^{-p_k^{(2)} \theta} \quad (5.25b)$$

where the  $a_k$  are the residues of the rightmost side of equation (5.23) at the poles  $p = -p_k^{(2)}$ . The values of these functions at  $\theta = 0$  read

$$r_{\text{eq}}(0) = \sum_{k \geq 1} a_k = e^{-\Lambda_{\text{eq}}} \quad (5.26a)$$

$$c_{\text{eq}}(0) = \sum_{k \geq 1} \frac{a_k}{p_k^{(2)}} = (f_0)_{\text{eq}}(1 - (f_0)_{\text{eq}}) = \frac{(1 - e^{-\Lambda_{\text{eq}}})(\Lambda_{\text{eq}} - 1 + e^{-\Lambda_{\text{eq}}})}{\Lambda_{\text{eq}}^2}. \quad (5.26b)$$

At infinite temperature ( $\Lambda_{\text{eq}} = 1$ ), the poles of equation (5.23) coincide with those of  $\hat{K}(p, \Lambda_{\text{eq}})$ , hence  $p_k^{(2)} = k$  for  $k \geq 1$ , and  $t_{\text{eq}}^{(2)} = 1$ . Again, as temperature decreases, the



**Figure 1.** Logarithmic plot of the relaxation times  $t_{\text{eq}}^{(1)}$  (mean energy) and  $t_{\text{eq}}^{(2)}$  (energy correlation and response functions), against inverse temperature  $\beta$ . Full curves: exact relaxation times (5.13), (5.24). Broken curves: low-temperature estimates (5.17), (5.28).

whole spectrum is continuously deformed, and the poles stay close to those of  $\hat{K}(p, \Lambda_{\text{eq}})$ :  $p_k^{(2)} \approx -k/\Lambda_{\text{eq}}$ . At low temperature, the relaxation time  $t_{\text{eq}}^{(2)}$  is exponentially divergent, while all the other characteristic times remain of order  $\Lambda_{\text{eq}}$ . Indeed, by expanding the denominator of the rightmost side of equation (5.23) to first order in  $p$ , and using again the expansion (5.14), we obtain

$$t_{\text{eq}}^{(2)} \approx \frac{(\Lambda_{\text{eq}} - 1)e^{\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}^2} (\Lambda_{\text{eq}}^2 e^{-\Lambda_{\text{eq}}} I(\Lambda_{\text{eq}}) + 1 - \Lambda_{\text{eq}}) \tag{5.27}$$

i.e.

$$t_{\text{eq}}^{(2)} \approx \frac{2e^{\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}} \left( 1 + \frac{2}{\Lambda_{\text{eq}}^2} + \dots \right) \approx \frac{2e^\beta}{\beta^2} \left( 1 + \frac{2 \ln \beta + 1}{\beta} + \dots \right). \tag{5.28}$$

Let us remark that the relaxation time  $t_{\text{eq}}^{(2)}$  of the energy correlation and response functions is roughly twice as large as the relaxation time  $t_{\text{eq}}^{(1)}$  of the mean energy. Indeed the ratio  $t_{\text{eq}}^{(2)}/t_{\text{eq}}^{(1)}$  is equal to two, both at infinite temperature ( $t_{\text{eq}}^{(1)} = \frac{1}{2}$ ,  $t_{\text{eq}}^{(2)} = 1$ ) and at zero temperature. We have in the low-temperature regime

$$\frac{t_{\text{eq}}^{(2)}}{t_{\text{eq}}^{(1)}} \approx 2 \left( 1 + \frac{1}{\Lambda_{\text{eq}}^2} + \dots \right) \approx 2 \left( 1 + \frac{1}{\beta^2} + \dots \right). \tag{5.29}$$

This ratio takes its maximal value 2.203 59 for  $\beta = 4.1986$ . Figure 1 shows the dependence on temperature of  $t_{\text{eq}}^{(1)}$  and  $t_{\text{eq}}^{(2)}$ , together with their low-temperature estimates (5.17), (5.28).

We have thus met two spectra of inverse characteristic times, the  $p_k^{(1)}$ , associated with the convergence of the mean energy to its equilibrium value, and the  $p_k^{(2)}$ , associated with the energy correlation and response functions at equilibrium. These spectra have a very similar dependence on temperature. Their low-temperature behaviour is characterized as follows: the relaxation times  $t_{\text{eq}}^{(1)}$  and  $t_{\text{eq}}^{(2)}$  become exponentially large in  $\Lambda_{\text{eq}}$  or in  $\beta$ , while the subleading characteristic times remain of order  $\Lambda_{\text{eq}}$ . In analogy with the phenomenology

of glassy dynamics, the slow modes corresponding to the time scale  $\theta \sim t_{\text{eq}}^{(1)} \sim t_{\text{eq}}^{(2)} \sim e^{\Lambda_{\text{eq}}}$  will be referred to as  $\alpha$  relaxation, while the fast modes corresponding to the microscopic time scale  $\theta \sim 1$  or  $\theta \sim \Lambda_{\text{eq}}$  will be referred to as  $\beta$  relaxation. This separation of the  $\alpha$  and  $\beta$  relaxation processes in a very clear-cut way is a remarkable feature of the present model.

#### 5.4. The energy correlation function

Let us now consider the low-temperature behaviour of the equilibrium correlation function  $c_{\text{eq}}(\theta)$ . Using again the expansion (5.14), we estimate the residue  $a_1$  as

$$a_1 \approx \frac{\Lambda_{\text{eq}} - 1}{\Lambda_{\text{eq}}^2} p_1^{(2)} \approx \frac{e^{-\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}^2 e^{-\Lambda_{\text{eq}}} I(\Lambda_{\text{eq}}) + 1 - \Lambda_{\text{eq}}} \approx \frac{e^{-\Lambda_{\text{eq}}}}{2} \left( 1 - \frac{1}{\Lambda_{\text{eq}}} - \frac{2}{\Lambda_{\text{eq}}^2} + \dots \right). \quad (5.30)$$

This expression shows that the sum of equation (5.26b) is overwhelmingly dominated by its first term ( $k = 1$ ), since the relative difference between the full sum and the first term is exponentially small. As a consequence, the energy correlation function at equilibrium is very close to being a pure decaying exponential, corresponding to  $\alpha$  relaxation:

$$C_{\text{eq}}(\theta) \approx e^{-\theta/t_{\text{eq}}^{(2)}} \quad (5.31)$$

with exponentially small corrections at low temperature.

#### 5.5. The energy response function

In contrast with the simple behaviour (5.31) of the correlation function, the response function  $r_{\text{eq}}(\theta)$  exhibits both  $\alpha$  and  $\beta$  relaxation processes at low temperature.

(1) In the regime of  $\beta$  relaxation ( $\theta \sim 1$ ), the response function at low temperature is obtained by expanding equation (5.12) for  $\Lambda_{\text{eq}}$  large but  $p$  finite:

$$\hat{K}(p, \Lambda_{\text{eq}}) = \frac{1}{p+1} - \frac{p}{(p+1)^3 \Lambda_{\text{eq}}} + \dots \quad (5.32)$$

hence

$$\hat{h}_0(p) \approx e^{-\Lambda_{\text{eq}}} \frac{p+1}{p(p+2)} \quad (5.33)$$

and

$$R_{\text{eq}}(\theta) \approx \frac{1}{2}(1 + e^{-2\theta}) \quad (5.34)$$

up to correction terms of order  $1/\Lambda_{\text{eq}}$ .

(2) In the regime of  $\alpha$  relaxation ( $\theta \sim t_{\text{eq}}^{(2)}$ ), the asymptotic fall-off of the response function reads

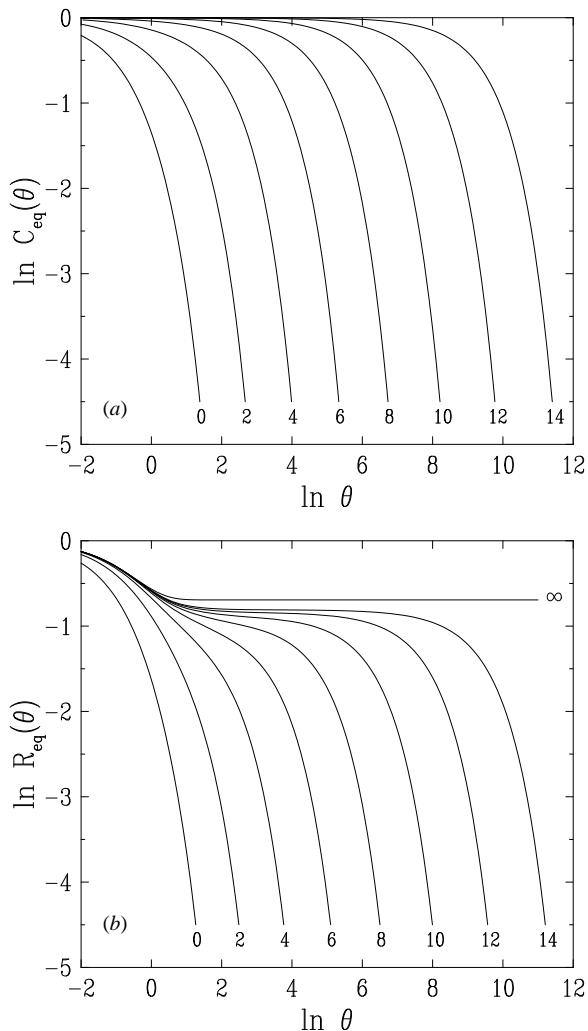
$$r_{\text{eq}}(\theta) \approx a_1 e^{-\theta/t_{\text{eq}}^{(2)}} \quad R_{\text{eq}}(\theta) \approx a_1 e^{\Lambda_{\text{eq}}} e^{-\theta/t_{\text{eq}}^{(2)}} \quad (5.35)$$

where  $a_1$  has been estimated in equation (5.30).

(3) In the crossover regime ( $1 \ll \theta \ll t_{\text{eq}}^{(2)}$ ), the normalized response function  $R_{\text{eq}}(\theta)$  exhibits a non-trivial  $\beta$ -to- $\alpha$  plateau value

$$R_{\text{pl}} \approx a_1 e^{\Lambda_{\text{eq}}} \approx \frac{1}{2} \left( 1 - \frac{1}{\Lambda_{\text{eq}}} + \dots \right) \approx \frac{1}{2} \left( 1 - \frac{1}{\beta} + \dots \right) \quad (5.36)$$

asymptotically equal to  $\frac{1}{2}$  at low temperature.



**Figure 2.** Log–log plot of (a) the normalized correlation function  $C_{\text{eq}}(\theta)$  and (b) the normalized response function  $R_{\text{eq}}(\theta)$  at equilibrium, as given by the exact expressions (5.25), against the time difference  $\theta = t - s$ , for various values of inverse temperature  $\beta$ , indicated on the curves.

Figure 2 shows log–log plots of (a) the normalized correlation function  $C_{\text{eq}}(\theta)$  and (b) the normalized response function  $R_{\text{eq}}(\theta)$ , for various values of the inverse temperature  $\beta$ . The different properties of these two functions, discussed above, are clearly apparent.

### 6. Non-equilibrium behaviour at low temperature

The equilibrium analysis of section 5 shows that, at low temperature, the model exhibits slow modes ( $\alpha$  relaxation, characterized by the time scales  $t_{\text{eq}}^{(1)}$  and  $t_{\text{eq}}^{(2)}$ ), well separated from fast ones ( $\beta$  relaxation, characterized by the microscopic time scale  $\sim 1$  or  $\Lambda_{\text{eq}}$ ). In this section, we show that this separation of modes at low temperature still holds out of equilibrium. This property is at the basis of the analysis presented below. The latter consists



of an adiabatic elimination of the fast modes (non-equilibrium  $\beta$  relaxation), generalizing the method used in [11]. We give an analytical treatment of the quantities of interest (mean energy, correlation and response functions), which becomes asymptotically exact for long times ( $s$  and  $t - s \gg 1$ , irrespectively of the relative positions of these variables with respect to the relaxation times  $t_{\text{eq}}^{(1)}$  and  $t_{\text{eq}}^{(2)}$ ) and low temperatures. This regime, hereafter called the (non-equilibrium)  $\alpha$  regime, encompasses in particular the aging regime ( $s$  and  $t - s \ll t_{\text{eq}}^{(1)}, t_{\text{eq}}^{(2)}$ ), and the regime of convergence to equilibrium ( $s \gg t_{\text{eq}}^{(1)}, t_{\text{eq}}^{(2)}$ ).

We begin the analysis by considering the difference

$$\tau(t) - \tau(u) = \int_u^t \frac{dv}{\Lambda(v)} \quad (6.1)$$

for  $t \gg 1$  but  $\varepsilon = t - u \sim 1$ . Since  $\Lambda(t)$  is varying slowly in the  $\alpha$  regime under consideration, we have  $d\Lambda(t)/dt \ll \Lambda(t)$ . Hence it is justified to perform a Taylor expansion of  $\Lambda(v)$  around  $v = t$ :

$$\tau(t) - \tau(u) = \int_u^t dv \left( \frac{1}{\Lambda(t)} + (t - v) \frac{d\Lambda(t)/dt}{\Lambda(t)^2} + \dots \right) = \frac{\varepsilon}{\Lambda(t)} + \frac{\varepsilon^2}{2} \frac{d\Lambda(t)/dt}{\Lambda(t)^2} + \dots \quad (6.2)$$

This expansion makes sense when  $\varepsilon$  is small with respect to the characteristic time of variation of  $\Lambda(t)$ . Note that performing a Taylor expansion around the lower bound  $v = u$  would lead to a result equivalent to equation (6.2). We have kept the first correction term, in order to show explicitly that it is of relative order  $(d\Lambda(t)/dt)/\Lambda(t) \ll 1$ . Indeed this quantity will turn out to be exponentially small in  $\Lambda(t)$  throughout the  $\alpha$  regime.

The result (6.2) can be used to derive the following estimates

$$\begin{aligned} D(t, u) &\approx \Lambda(t)(1 - e^{-\varepsilon/\Lambda(t)}) \\ \mathcal{K}(0, t, u) &\approx \exp(-\varepsilon/\Lambda(t) - \Lambda(t)(1 - e^{-\varepsilon/\Lambda(t)})) = K(\varepsilon, \Lambda(t)) \end{aligned} \quad (6.3)$$

where  $K(\varepsilon, \Lambda(t))$  is obtained by replacing  $\Lambda_{\text{eq}}$  by  $\Lambda(t)$  in the expressions (5.12). As mentioned above, the approximate results (6.2), (6.3) hold for  $t \gg 1$  but  $\varepsilon = t - u \sim 1$ . It will, however, be legitimate to employ them for an arbitrary time difference  $\varepsilon$ , since the quantities  $D(t, u) - \Lambda(t)$  and  $\mathcal{K}(0, t, u)$  decay exponentially with  $\varepsilon$  anyhow.

### 6.1. Mean energy

Considering now the integral equation (3.9a) for  $\Lambda(t)$ , we perform a Taylor expansion of  $Y_f(u)$  near  $u = t$ . It can be checked *a posteriori* that it is consistent to keep the first two terms of this expansion. Neglecting furthermore quantities which are exponentially small in  $t$ , we transform equation (3.9a) into the differential equation

$$f_0(t) \approx e^{-\Lambda(t)} + J_0(t)Y_f(t) + J_1(t) \frac{dY_f(t)}{dt} + \dots \quad (6.4)$$

with

$$\begin{aligned}
 J_0(t) &= \int_0^t du \mathcal{K}(0, t, u) \\
 &\approx \int_0^\infty d\varepsilon K(\varepsilon, \Lambda(t)) = \hat{K}(0, \Lambda(t)) = 1 - e^{-\Lambda(t)} \\
 J_1(t) &= \int_0^t du (u - t)\mathcal{K}(0, t, u) \\
 &\approx - \int_0^\infty d\varepsilon \varepsilon K(\varepsilon, \Lambda(t)) = \left. \frac{d\hat{K}(p, \Lambda(t))}{dp} \right|_{p=0} = -\Lambda(t)e^{-\Lambda(t)} I(\Lambda(t)).
 \end{aligned}
 \tag{6.5}$$

By inserting the estimates (6.5) into equation (6.4), using equations (2.6a) and (3.3a), and consistently neglecting terms of relative order  $e^{-\Lambda(t)}$  or  $e^{-\beta}$ , we obtain

$$\frac{d\Lambda(t)}{dt} \approx A(\Lambda(t), \Lambda_{\text{eq}})
 \tag{6.6}$$

with

$$A(\Lambda, \Lambda_{\text{eq}}) = \frac{1}{I(\Lambda)} \left( 1 - \frac{(\Lambda - 1)e^\Lambda}{(\Lambda_{\text{eq}} - 1)e^{\Lambda_{\text{eq}}}} \right).
 \tag{6.7}$$

The evolution equation (6.6) for  $\Lambda(t)$  describes the relaxation of energy throughout the  $\alpha$  regime. Let us consider the following cases.

(1) At zero temperature, and more generally in the aging regime ( $t \ll t_{\text{eq}}^{(1)}$ ), we recover the result [11]:

$$\frac{d\Lambda}{dt} \approx \frac{1}{I(\Lambda)}
 \tag{6.8}$$

hence

$$t - t_0 \approx \int_0^\Lambda d\Lambda' I(\Lambda') = \int_0^1 \frac{dz}{z^2} (e^{\Lambda z} - 1 - \Lambda z) = \sum_{n \geq 1} \frac{\Lambda^{n+1}}{n(n+1)!} \approx \frac{e^\Lambda}{\Lambda} \sum_{\ell \geq 0} \frac{(\ell+1)!}{\Lambda^\ell}
 \tag{6.9}$$

or else

$$\Lambda(t) \approx \ln t + \ln \ln t + \frac{\ln \ln t - 2}{\ln t} + \dots
 \tag{6.10}$$

It is natural to choose the constant  $t_0$  in order to fulfil the initial condition  $\Lambda(t = 0) = 1$ , yielding  $t_0 = -0.59962$ . Anyhow,  $t_0$  only brings corrections to the scaling law (6.10) which is exponentially small in  $\Lambda(t)$ , i.e. comparable to the corrections terms which have been neglected during the analysis.

(2) At finite (low) temperature, thermal equilibrium is reached for long enough times. The convergence of  $\Lambda(t)$  towards the equilibrium value  $\Lambda_{\text{eq}}$  is exponential:

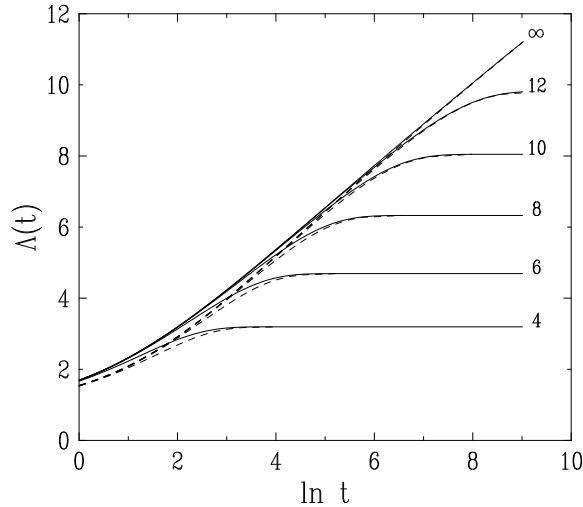
$$\Lambda_{\text{eq}} - \Lambda(t) \approx a(\Lambda_{\text{eq}})e^{-t/t_{\text{eq}}^{(1)}}.
 \tag{6.11}$$

The relaxation time  $t_{\text{eq}}^{(1)}$  reads

$$t_{\text{eq}}^{(1)} \approx - \frac{1}{\partial A(\Lambda, \Lambda_{\text{eq}})/\partial \Lambda|_{\Lambda=\Lambda_{\text{eq}}}} \approx \frac{\Lambda_{\text{eq}} - 1}{\Lambda_{\text{eq}}} I(\Lambda_{\text{eq}})
 \tag{6.12}$$

in agreement with the result (5.16) at equilibrium, while the prefactor  $a(\Lambda_{\text{eq}})$  is given by

$$t_{\text{eq}}^{(1)} \ln \frac{a(\Lambda_{\text{eq}})}{\Lambda_{\text{eq}}} = \int_0^{\Lambda_{\text{eq}}} d\Lambda \left( \frac{1}{A(\Lambda, \Lambda_{\text{eq}})} - \frac{t_{\text{eq}}^{(1)}}{\Lambda_{\text{eq}} - \Lambda} \right)
 \tag{6.13}$$



**Figure 3.** Plot of  $\Lambda(t)$  against  $\ln t$ , for various values of inverse temperature  $\beta$ , indicated on the curves. Full curves: exact values, obtained by numerical integration of the dynamical equations (2.5). Broken curves: low-temperature estimates (6.6).

yielding, after some algebra,  $a(\Lambda_{\text{eq}}) \approx 1$  for  $\Lambda_{\text{eq}}$  large.

(3) When time  $t$  is comparable to the relaxation time  $t_{\text{eq}}^{(1)}$ ,  $\Lambda(t)$  changes over from the zero-temperature logarithmic growth (6.10) to the exponential convergence (6.11) towards the equilibrium value  $\Lambda_{\text{eq}}$ . The scaling behaviour throughout this crossover can be described, to leading order in  $1/\Lambda$ , by approximating equation (6.7) as

$$A(\Lambda, \Lambda_{\text{eq}}) \approx \Lambda(e^{-\Lambda} - e^{-\Lambda_{\text{eq}}}). \quad (6.14)$$

This equation can be integrated, to leading order in  $1/\Lambda(t)$  and  $1/\Lambda_{\text{eq}}$ . Using the definition (6.12), we are left with the explicit interpolation formula

$$\frac{t}{t_{\text{eq}}^{(1)}} \approx -\frac{\Lambda_{\text{eq}}}{\Lambda(t)} \ln(1 - e^{\Lambda(t) - \Lambda_{\text{eq}}}). \quad (6.15)$$

Figure 3 illustrates the accuracy of the low-temperature analysis in the  $\alpha$  regime. Exact values for  $\Lambda(t)$ , obtained by numerical integration of the dynamical equations (2.5), are found to nicely agree with the prediction of the evolution equation (6.6), for various (large) values of the inverse temperature  $\beta$ .

## 6.2. The energy correlation and response functions: generalities

In analogy with the derivation of equation (6.4), we transform equations (3.9b) and (3.9c) into the following differential equations

$$\begin{aligned} g_0(t, s) &\approx e^{-\Lambda(t)} + L_0(t, s)Y_g(t, s) + L_1(t, s)\frac{\partial Y_g(t, s)}{\partial t} + \dots \\ h_0(t, s) &\approx L_0(t, s)Y_h(t, s) + L_1(t, s)\frac{\partial Y_h(t, s)}{\partial t} + \dots \end{aligned} \quad (6.16)$$

with

$$L_0(t, s) = \int_s^t du \mathcal{K}(0, t, u) \quad L_1(t, s) = \int_s^t du (u - t)\mathcal{K}(0, t, u). \quad (6.17)$$

These integrals differ from  $J_0(t)$  and  $J_1(t)$  of equation (6.4) only through the values of their lower bounds. They are therefore given by the expressions (6.5), again up to exponentially small corrections.

By inserting the estimates (6.5) into equation (6.16), using equation (3.3b) and similar expressions for  $Y_h$ , and again consistently neglecting terms of relative order  $e^{-\Lambda}$  or  $e^{-\beta}$ , we obtain the homogeneous differential equations

$$\frac{1}{c(t, s)} \frac{\partial c(t, s)}{\partial t} \approx \frac{1}{r(t, s)} \frac{\partial r(t, s)}{\partial t} \approx -B(\Lambda(t), \Lambda_{\text{eq}}) \quad (6.18)$$

with

$$B(\Lambda, \Lambda_{\text{eq}}) = \frac{A(\Lambda, \Lambda_{\text{eq}}) + \Lambda^2 \left( e^{-\Lambda} + \frac{e^{-\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}} - 1} \right)}{\Lambda(\Lambda^2 e^{-\Lambda} I(\Lambda) + 1 - \Lambda)}. \quad (6.19)$$

Changing the time variables from  $s$  and  $t$  to  $\Lambda(s)$  and  $\Lambda(t)$ , we have the alternative form

$$\frac{1}{c(t, s)} \frac{\partial c(t, s)}{\partial \Lambda(t)} \approx \frac{1}{r(t, s)} \frac{\partial r(t, s)}{\partial \Lambda(t)} \approx -\alpha(\Lambda(t), \Lambda_{\text{eq}}) \quad (6.20)$$

with

$$\alpha(\Lambda, \Lambda_{\text{eq}}) = \frac{B(\Lambda, \Lambda_{\text{eq}})}{A(\Lambda, \Lambda_{\text{eq}})}. \quad (6.21)$$

These evolution equations describe the non-equilibrium relaxation of the correlation and response functions of the energy throughout the  $\alpha$  regime at low temperature, as we shall see in more detail.

### 6.3. The energy correlation function

The correlation function  $c(t, s)$  can be directly estimated from equation (6.20) with the initial value  $c(s, s) = f_0(s)(1 - f_0(s))$ , or  $C(s, s) = 1$ . Indeed, we know from the analysis of the equilibrium properties that this function varies very little in the  $\beta$  regime ( $t - s \sim 1$ ). We thus obtain

$$C(t, s) \approx \exp \left( - \int_{\Lambda(s)}^{\Lambda(t)} d\Lambda' \alpha(\Lambda', \Lambda_{\text{eq}}) \right). \quad (6.22)$$

This result can be recast into a multiplicative scaling law:

$$C(t, s) \approx \frac{\Phi(\Lambda(s), \Lambda_{\text{eq}})}{\Phi(\Lambda(t), \Lambda_{\text{eq}})} \quad (6.23)$$

with

$$\Phi(\Lambda, \Lambda_{\text{eq}}) = \exp \left( \int_{\Lambda_0}^{\Lambda} d\Lambda' \alpha(\Lambda', \Lambda_{\text{eq}}) \right). \quad (6.24)$$

An interesting interpretation of equation (6.23) consists in introducing an effective waiting time

$$s_{\text{eff}} = \frac{\Phi(\Lambda(s), \Lambda_{\text{eq}})}{d\Phi(\Lambda(s), \Lambda_{\text{eq}})/ds} = \frac{1}{B(\Lambda(s), \Lambda_{\text{eq}})} \quad (6.25)$$

along the lines of [6]. The correlation function and the effective waiting time exhibit several regimes.

(1) At zero temperature, and more generally in the aging regime ( $s \ll t_{\text{eq}}^{(2)}$ ), we have

$$\alpha(\Lambda, \infty) = \frac{\Lambda^2 e^{-\Lambda} I(\Lambda) + 1}{\Lambda(\Lambda^2 e^{-\Lambda} I(\Lambda) + 1 - \Lambda)} \approx \frac{1}{2} \left( 1 + \frac{1}{\Lambda} - \frac{2}{\Lambda^2} + \dots \right) \quad (6.26)$$

hence the approximately square-root behaviour

$$\Phi(\Lambda(s), \infty) \approx c_0(\Lambda(s)e^{\Lambda(s)})^{1/2} \left(1 - \frac{1}{\Lambda(s)} + \dots\right) \approx c_0 s^{1/2} \ln s \left(1 + \frac{\ln \ln s - 2}{\ln s} + \dots\right). \quad (6.27)$$

The absolute prefactor  $c_0$  depends on the initial conditions, according to

$$c_0 = \exp\left(\int_{\Lambda_0}^{\infty} d\Lambda \left[\alpha(\Lambda, \infty) - \frac{1}{2} \left(1 + \frac{1}{\Lambda}\right)\right]\right) \quad (6.28)$$

with the natural choice  $\Lambda_0 = 1$  yielding  $c_0 = 6.233\ 67$ . The effective waiting time reads

$$s_{\text{eff}} \approx 2s \left(1 - \frac{2}{\Lambda(s)} + \dots\right) \approx 2s \left(1 - \frac{2}{\ln s} + \dots\right). \quad (6.29)$$

(2) At finite (low) temperature and for long enough times, the scaling function  $\Phi$  blows up exponentially, as

$$\Phi(\Lambda(s), \Lambda_{\text{eq}}) \approx c_0 b(\Lambda_{\text{eq}}) e^{s/t_{\text{eq}}^{(2)}} \quad (6.30)$$

with

$$s_{\text{eff}} = t_{\text{eq}}^{(2)} \approx \frac{1}{B(\Lambda_{\text{eq}}, \Lambda_{\text{eq}})} = \frac{(\Lambda_{\text{eq}} - 1)e^{\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}^2} (\Lambda_{\text{eq}}^2 e^{-\Lambda_{\text{eq}}} I(\Lambda_{\text{eq}}) + 1 - \Lambda_{\text{eq}}) \quad (6.31)$$

in agreement with equation (5.27). The prefactor  $b(\Lambda_{\text{eq}})$  can be given a convergent integral expression, similar to equation (6.13).

(3) In the crossover region, when  $s$  is comparable to the relaxation times  $t_{\text{eq}}^{(1)}$  and  $t_{\text{eq}}^{(2)}$ , the scaling function  $\Phi(\Lambda(s), \Lambda_{\text{eq}})$  changes over from the approximately square-root behaviour (6.27) to the exponential behaviour (6.30). The scaling behaviour in this crossover can be described, to leading order in  $1/\Lambda$ , by approximating equation (6.21) as

$$\alpha(\Lambda, \Lambda_{\text{eq}}) \approx \frac{1}{2(1 - e^{\Lambda - \Lambda_{\text{eq}}})}. \quad (6.32)$$

This equation can be integrated, to leading order in  $1/\Lambda(s)$  and  $1/\Lambda_{\text{eq}}$ . We thus obtain

$$\Phi(\Lambda(s), \Lambda_{\text{eq}}) \approx (e^{-\Lambda(s)} - e^{-\Lambda_{\text{eq}}})^{-1/2} \quad (6.33)$$

implying the estimate  $\ln b(\Lambda_{\text{eq}}) \approx \Lambda_{\text{eq}}/2$  for  $\Lambda_{\text{eq}}$  large. Accordingly, the effective waiting time departs from its linear growth (6.29) in  $s$  to saturate to the value of the relaxation time  $t_{\text{eq}}^{(2)}$ . This quantity admits the simple expression

$$s_{\text{eff}} \approx \frac{2e^{\Lambda(s)}}{\Lambda(s)} \quad (6.34)$$

again to leading order in  $1/\Lambda(s)$ , throughout the crossover. The above results allow us to describe the behaviour of the correlation function in the  $\alpha$  regime, by inserting into the scaling law (6.23) the estimates (6.27), (6.30) and (6.33) for each of the times  $s$  and  $t$ . The absolute prefactors  $c_0$  cancel out, so that the scaling law (6.23) does not depend on the initial conditions, in agreement with the fact that only large times enter the expression (6.22). In particular, in the aging regime ( $s$  and  $t \ll t_{\text{eq}}^{(2)}$ ), both  $\Phi$  functions have the behaviour (6.27), so that we obtain

$$C(t, s) \approx \left(\frac{s}{t}\right)^{1/2} \frac{\ln s}{\ln t} \quad (6.35)$$

recovering the result of the zero-temperature analysis [11], recalled in equation (1.1). In the converse situation ( $s, t \gg t_{\text{eq}}^{(2)}$ ), both  $\Phi$  functions have the exponential behaviour (6.30), so that the equilibrium correlation function (5.31) is recovered.

6.4. The energy response function

The response function  $r(t, s)$  is expected to behave differently from the correlation function  $c(t, s)$ , and especially to exhibit an appreciable variation in the  $\beta$  regime ( $\theta = t - s \sim 1$ ), in analogy with its equilibrium behaviour, described in section 5.

The variation of  $r(t, s)$  throughout the non-equilibrium  $\beta$  regime can be determined by considering  $h_0(t, s)$  as a function of  $\theta$ , and by solving equation (3.9c) accordingly, by means of a Laplace transform, as in section 5, up to the following differences: take  $\Lambda$  equal to  $\Lambda(s)$ , and not to its equilibrium value  $\Lambda_{\text{eq}}$ ; consider  $f_0(t)$  and  $f_1(t)$  as variable quantities, given by the dynamical equations (2.5). With these assumptions, closed equations for the Laplace transforms  $\hat{h}_0(p)$ ,  $\hat{h}_1(p)$  and  $\hat{Y}_h(p)$  can be obtained, yielding after some algebra

$$r(t, s) \approx \mu(s) f_0(s) R(\Lambda(s), \theta) \tag{6.36}$$

with

$$\mu(s) f_0(s) \approx \frac{(\Lambda(s) - 1)(\Lambda(s)^2 e^{-\beta} + A(\Lambda(s), \Lambda_{\text{eq}}))}{\Lambda(s)^2} \tag{6.37}$$

and where  $R(\Lambda(s), \theta)$  is obtained by replacing  $\Lambda_{\text{eq}}$  by  $\Lambda(s)$  in the expression of the normalized response function  $R_{\text{eq}}(\theta) = e^{\Lambda_{\text{eq}}} r_{\text{eq}}(\theta)$  at equilibrium, with  $r_{\text{eq}}(\theta)$  being given in equation (5.25a). The result (6.36) holds at low temperature, and for  $s \gg 1$  and  $\theta$  finite.

The response function exhibits a  $\beta$ -to- $\alpha$  plateau for  $1 \ll \theta \ll t_{\text{eq}}^{(2)}$ , with a plateau value

$$r_{\text{pl}}(s) \approx \mu(s) f_0(s) R_{\text{pl}}(\Lambda(s)) \tag{6.38}$$

with (see equations (5.30) and (5.36))

$$R_{\text{pl}}(\Lambda(s)) \approx \frac{1}{\Lambda(s)^2 e^{-\Lambda(s)} I(\Lambda(s)) + 1 - \Lambda(s)}. \tag{6.39}$$

The result (6.38) interpolates between the equilibrium value  $r_{\text{pl}}(s) = a_1$  for  $s \gg t_{\text{eq}}^{(2)}$  and the zero-temperature value

$$r_{\text{pl}}(s) \approx \frac{\Lambda(s) - 1}{\Lambda(s)^2 I(\Lambda(s)) (\Lambda(s)^2 e^{-\Lambda(s)} I(\Lambda(s)) + 1 - \Lambda(s))} \tag{6.40}$$

for  $s \ll t_{\text{eq}}^{(2)}$ . The plateau value (6.38) is the appropriate initial condition to be inserted into equation (6.18), in order to describe the non-equilibrium  $\alpha$  relaxation of the response function. We thus obtain

$$r(t, s) \approx r_{\text{pl}}(s) C(t, s). \tag{6.41}$$

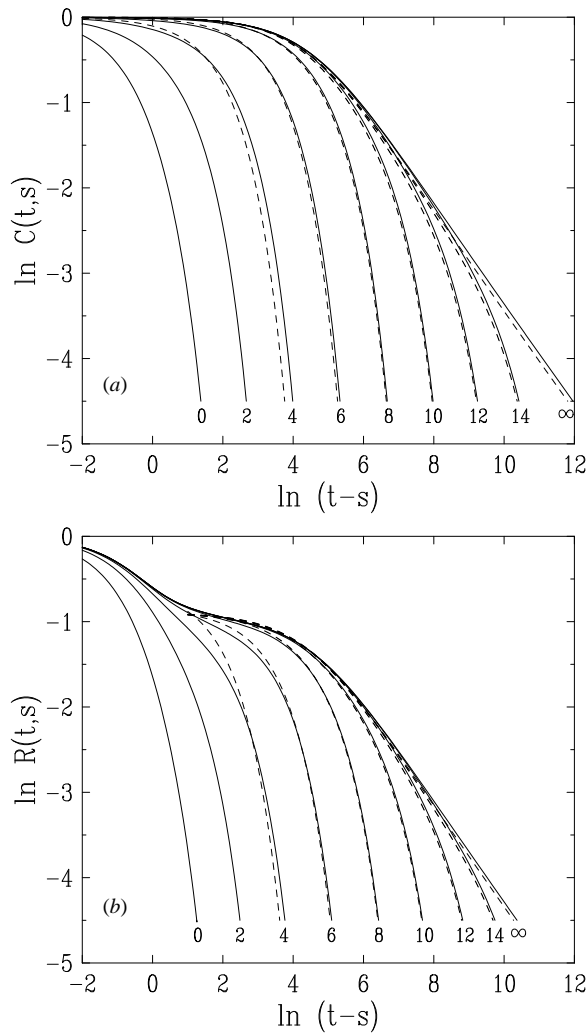
The above discussion of the scaling behaviour of the correlation function  $C(t, s)$  applies to the response function  $r(t, s)$  throughout the low-temperature  $\alpha$  regime.

Figure 4 shows log-log plots of the normalized correlation function and response function, for various values of the inverse temperature  $\beta$ . The data again nicely agree with the predictions (6.23) and (6.41) of the low-temperature analysis in the  $\alpha$  regime.

6.5. The fluctuation-dissipation ratio

The above results yield the following description of the non-equilibrium behaviour of the fluctuation-dissipation ratio  $X(t, s)$ . In the  $\beta$  regime, we can approximate  $g_0(t, s)$  by unity. Setting again  $\theta = t - s$ , and using equation (6.36), we obtain

$$X(t, s) \approx \frac{\Lambda(s)^3 \mu(s) f_0(s) R(\Lambda(s), \theta)}{\Lambda(s)^3 \mu(s) f_0(s) R(\Lambda(s), \theta) + A(\Lambda(s), \Lambda_{\text{eq}})}. \tag{6.42}$$



**Figure 4.** Log-log plot of (a) the normalized correlation function  $C(t, s)$  and (b) the normalized response function  $R(t, s)$ , against the time difference  $t - s$ , for a waiting time  $s = 100$ , and various values of inverse temperature  $\beta$ , indicated on the curves. Full curves: exact values, obtained by numerical integration of the dynamical equations (2.14), (2.26). Broken curves: predictions (6.23), (6.41) of the low-temperature analysis in the  $\alpha$  regime.

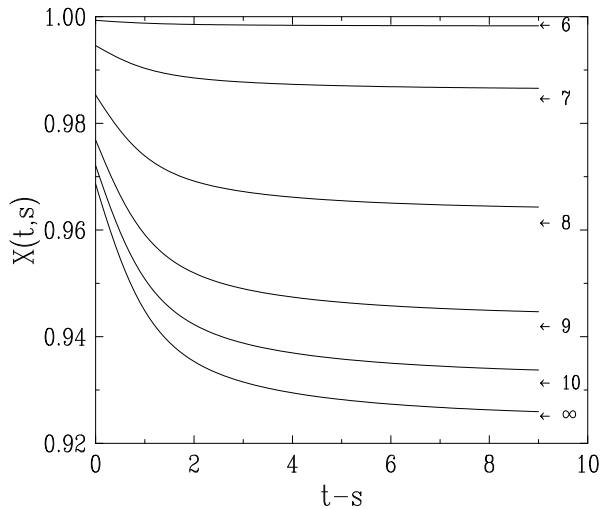
Hence  $X(t, s)$  also exhibits a  $\beta$ -to- $\alpha$  plateau for  $1 \ll \theta \ll t_{\text{eq}}^{(2)}$ , with a plateau value

$$X_{\text{pl}}(s) \approx \frac{\Lambda(s)^3 \mu(s) f_0(s) R_{\text{pl}}(\Lambda(s))}{\Lambda(s)^3 \mu(s) f_0(s) R_{\text{pl}}(\Lambda(s)) + A(\Lambda(s), \Lambda_{\text{eq}})}. \quad (6.43)$$

This result interpolates between the equilibrium value  $X_{\text{pl}}(s) = 1$  for  $s \gg t_{\text{eq}}^{(2)}$  and the value

$$X_{\text{pl}}(s) \approx \frac{\Lambda(s)(\Lambda(s) - 1)}{\Lambda(s)^2 e^{-\Lambda(s)} I(\Lambda(s)) + (\Lambda(s) - 1)^2} \approx 1 - \frac{2}{\Lambda(s)^2} + \dots \approx 1 - \frac{2}{(\ln s)^2} + \dots \quad (6.44)$$

at zero temperature, and more generally for  $s \ll t_{\text{eq}}^{(2)}$ .



**Figure 5.** Plot of the fluctuation–dissipation ratio  $X(t, s)$  against the time difference  $t - s$ , for a waiting time  $s = 100$ , and various values of inverse temperature  $\beta$ , indicated on the curves. Full curves: exact values, obtained by numerical integration of the dynamical equations (2.14), (2.26). Arrows: low-temperature estimates (6.43) for the plateau value  $X_{\text{pl}}$ .

Finally, the evolution equations (6.18) and (6.20) imply that the fluctuation–dissipation ratio  $X(t, s)$  remains equal to  $X_{\text{pl}}(s)$  for all subsequent times ( $\theta = t - s \gg 1$ ). In other words, the evolution of the fluctuation–dissipation ratio does not couple at all to the slow dynamics of the model in the  $\alpha$  regime. Figure 5 illustrates this behaviour.

## 7. Discussion

The present work is a continuation of previous studies [9, 11] of the so-called Backgammon model, introduced in the context of the dynamics of glasses [7]. We have focused our attention on the mean energy of the model, its local (or diagonal, see below) correlation and response functions at finite temperature, and the associated fluctuation–dissipation ratio. The analysis of equilibrium properties (section 5) has revealed a remarkable feature of the model, namely the possibility of separating in a controlled way the slow modes ( $\alpha$  relaxation) from the fast ones ( $\beta$  relaxation). This property is at the basis of the analytical treatment of the dynamics (section 6), which becomes asymptotically exact in the regime of non-equilibrium  $\alpha$  relaxation: low temperatures and long times ( $s$  and  $t - s \gg 1$ ). This approach consists in eliminating the fast degrees of freedom, corresponding to  $\beta$  relaxation. It allows a quantitative description of the aging regime (correlation and response decay as a power law, the effective waiting time grows linearly), of the convergence to equilibrium (correlation and response decay exponentially with characteristic time  $t_{\text{eq}}^{(2)}$  of the equilibrium situation, the effective waiting time saturates to  $t_{\text{eq}}^{(2)}$ ), and of the crossover behaviour between them. The present method is an extension of the approach developed at zero temperature [11], interpreted there as an adiabatic approximation. It is worth noticing that the physical picture of  $\alpha$  and  $\beta$  relaxation was absent from this zero-temperature analysis.

The multiplicative scaling law (6.23) of the energy correlation function in the  $\alpha$  regime is one of the main results of this work. It is consistent with the description of non-equilibrium glassy dynamics given, for example, in [6]. Especially, for any three (long enough) times,



we have  $C(t_1, t_3) = F(C(t_1, t_2), C(t_2, t_3))$ , with  $F(u, v) = uv$ . The associated fixed-point equation  $u = F(u, u)$  only has the trivial solutions  $u = 0$  and  $u = 1$ . This simplistic behaviour reflects that the correlation function has no non-trivial plateau value. An investigation of the density correlation and response functions, and of the associated fluctuation–dissipation ratio, will be the subject of a separate paper [13]. The density correlation function does exhibit a non-trivial plateau value [12].

In this work we concentrated on the diagonal parts of the correlation and response functions of energy, which involve the same box at two different times  $s$  and  $t$ . These diagonal contributions are known to be the leading ones in disordered systems, such as spin glasses. In the present model, however, non-diagonal correlations, involving say box number 1 at time  $s$  and box number 2 at time  $t$ , are not negligible in general. Let us take the example of equal-time equilibrium correlations of the energy. The diagonal correlation  $\langle E_1^2 \rangle - \langle E_1 \rangle^2 = (f_0)_{\text{eq}}(1 - (f_0)_{\text{eq}})$  yields a contribution  $C_{\text{eq}}^{\text{diag}} = \beta^2 (f_0)_{\text{eq}}(1 - (f_0)_{\text{eq}}) \approx \beta$  to the low-temperature specific heat per box. On the other hand, the total specific heat at low temperature, given in equation (C.8), reads  $C_{\text{eq}} \approx 1$ . We have therefore  $C_{\text{eq}} \ll C_{\text{eq}}^{\text{diag}}$ . This demonstrates the role of non-diagonal correlations, which screen the diagonal ones almost perfectly in this example.

The present work can also be extended in another direction. In [9] three variants of the Backgammon model, with different dynamical rules, were introduced and investigated at zero temperature. The model studied in the present work is model A, the other two being models B and C. We wish to emphasize that only model A has the property that the spectrum of relaxation times at equilibrium can be split in a controlled way into an  $\alpha$  and a  $\beta$  component. We also recall that only model A possesses entropic traps and a very slow zero-temperature dynamics ( $\Lambda(t) \sim \ln t$ ), while  $\Lambda(t)$  obeys a power-law growth for models B and C [9]. These two properties therefore seem to go hand in hand. Finally, we would like to mention a generalization of model B which undergoes a condensation transition at finite temperature [15].

## Acknowledgments

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## Appendix A. The derivation of dynamical equations

### A.1. The case of $f_k$ , $g_k$ , and $h_k$

The rules defining the dynamics of the model are as follows. For a system of  $M$  boxes and  $N$  particles, during every time step  $dt = 1/N$ :

- a particle is drawn at random, which determines a departure box  $d$  which is thus chosen with probability  $\pi_d = N_d/N$ , since it contains  $N_d$  particles;
- an arrival box  $a$ , different from  $d$ , is drawn at random with uniform probability  $\pi_a = 1/(M - 1)$ ;
- the move is accepted according to the Metropolis rule.

A configuration of the system is given by the occupation numbers of the boxes:  $\mathcal{C} = \{N_1, N_2, \dots, N_M\}$ , with  $\sum_i N_i = N$ . We first determine the transition rates appearing in the

master equation

$$\frac{d}{dt}P(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} \mathcal{M}(\mathcal{C}|\mathcal{C}')P(\mathcal{C}') - \sum_{\mathcal{C}'' \neq \mathcal{C}} \mathcal{M}(\mathcal{C}''|\mathcal{C})P(\mathcal{C}). \tag{A.1}$$

Let us consider the rate  $\mathcal{M}(\mathcal{C}''|\mathcal{C})$  for definiteness. We use the following notation for the initial configuration:  $\mathcal{C} = \{N_1, \dots, N_d, \dots, N_a, \dots, N_M\} = \{N_d, N_a\}$ . Then the final configuration is  $\mathcal{C}'' = \{N_d - 1, N_a + 1\}$ . The Metropolis acceptance rate reads

$$W(N_d - 1, N_a + 1|N_d, N_a) = \mathcal{W}(\Delta S) = \min(1, e^{-\Delta S}) \tag{A.2}$$

where the change in action is

$$\Delta S = S(N_d - 1, N_a + 1) - S(N_d, N_a) = -\beta(\delta_{N_d,1} - \delta_{N_a,0}). \tag{A.3}$$

The possible transitions to be considered are

$$\begin{aligned} \mathcal{C} = \{N_d > 1, N_a > 0\} & \quad \Delta S = 0 & \quad W(N_d - 1, N_a + 1|N_d, N_a) = 1 \\ \mathcal{C} = \{N_d > 1, N_a = 0\} & \quad \Delta S = \beta & \quad W(N_d - 1, N_a + 1|N_d, N_a) = e^{-\beta} \\ \mathcal{C} = \{N_d = 1, N_a > 0\} & \quad \Delta S = -\beta & \quad W(N_d - 1, N_a + 1|N_d, N_a) = 1 \\ \mathcal{C} = \{N_d = 1, N_a = 0\} & \quad \Delta S = 0 & \quad W(N_d - 1, N_a + 1|N_d, N_a) = 1. \end{aligned} \tag{A.4}$$

The transition rate  $\mathcal{M}(\mathcal{C}''|\mathcal{C})$  is thus given by

$$\mathcal{M}(\mathcal{C}''|\mathcal{C}) = \mathcal{M}(N_d - 1, N_a + 1|N_d, N_a) = \frac{1}{dt} \pi_d \pi_a W(N_d - 1, N_a + 1|N_d, N_a). \tag{A.5}$$

We now consider the reduced configuration characterized by the occupation number of box number 1 alone:  $\mathcal{C} = \{N_1(t)\} = \{k\}$ . The transition rates  $\mathcal{M}(\mathcal{C}''|\mathcal{C}) = \mathcal{M}(k \pm 1|k)$  respectively correspond to the case where box number 1 is the arrival box ( $a = 1$ : upper sign) and the departure box ( $d = 1$ : lower sign). They are obtained by summing equation (A.5) over all the boxes. In the thermodynamic limit ( $M, N \rightarrow \infty$ , with a fixed density  $\rho = N/M$ ), we obtain†

$$\begin{aligned} \mathcal{M}(k + 1|k) &= \frac{1}{dt} \sum_d \pi_d \pi_1 W(N_d - 1, N_1 + 1|N_d, N_1) = \rho \quad (k \geq 1) \\ \mathcal{M}(k - 1|k) &= \frac{1}{dt} \sum_a \pi_1 \pi_a W(N_1 - 1, N_a + 1|N_1, N_a) = k(1 - f_0 + e^{-\beta} f_0) \quad (k \geq 2) \\ \mathcal{M}(1|0) &= \frac{1}{dt} \sum_d \pi_d \pi_1 W(N_d - 1, N_1 + 1|N_d, N_1) = f_1 + e^{-\beta}(\rho - f_1) \\ \mathcal{M}(0|1) &= \frac{1}{dt} \sum_a \pi_1 \pi_a W(N_1 - 1, N_a + 1|N_1, N_a) = 1. \end{aligned} \tag{A.6}$$

The rates  $\mathcal{M}(\mathcal{C}|\mathcal{C}') = \mathcal{M}(k|k \pm 1)$  are also given by these expressions.

With the notation  $\mathcal{M}_{k\ell}[f_0(t), f_1(t)] = \mathcal{M}(k|\ell)$ , and setting  $\rho = 1$ , we obtain the dynamical equations (2.5) and (2.7), i.e.

$$\frac{df_k(t)}{dt} = \sum_{\ell \geq 0} \mathcal{M}_{k\ell}[f_0(t), f_1(t)] f_\ell(t). \tag{A.7}$$

The  $g_k(t, s)$  obey the similar equation (2.14), since the time variable  $s$  plays the role of a parameter in the dynamics. Finally, the  $h_k(t, s)$  obey equation (2.26), as explained in section 2.

† In this limit, it does not matter whether the boxes which are summed upon are assumed to be different or not from box number 1.

### A.2. The case of $f_k^{\beta_1}$

The only change with respect to the previous case comes from the fact that the local inverse temperature of box number 1 is now  $\beta_1(t)$ . The action therefore reads

$$S = \beta_1(t)E_1 + \beta \sum_{i=2}^M E_i = -\beta_1(t)\delta_{N_1,0} - \beta \sum_{i=2}^M \delta_{N_i,0} \quad (\text{A.8})$$

hence

$$\Delta S = S(N_d - 1, N_a + 1) - S(N_d, N_a) = \beta_a \delta_{N_a,0} - \beta_d \delta_{N_d,1}. \quad (\text{A.9})$$

The possible transitions to be considered are

$$\begin{aligned} \mathcal{C} = \{N_d > 1, N_a > 0\} & \quad \Delta S = 0 & \quad W(N_d - 1, N_a + 1 | N_d, N_a) = 1 \\ \mathcal{C} = \{N_d > 1, N_a = 0\} & \quad \Delta S = \beta_a & \quad W(N_d - 1, N_a + 1 | N_d, N_a) = e^{-\beta_a} \\ \mathcal{C} = \{N_d = 1, N_a > 0\} & \quad \Delta S = -\beta_d & \quad W(N_d - 1, N_a + 1 | N_d, N_a) = 1 \\ \mathcal{C} = \{N_d = 1, N_a = 0\} & \quad \Delta S = \beta_a - \beta_d & \quad W(N_d - 1, N_a + 1 | N_d, N_a) = \mathcal{W}(\Delta S). \end{aligned} \quad (\text{A.10})$$

Pursuing the derivation as above, we obtain the dynamical equations (2.19).

### Appendix B. The solutions of equations (3.2) by the method of characteristics

We recall how to solve, using the method of characteristics, a partial differential equation of the form

$$\frac{\partial}{\partial t} G(x, t, s) = (x - 1) \left( G(x, t, s) - \frac{1}{\Lambda(t)} \frac{\partial}{\partial x} G(x, t, s) - Y(t, s) \right) \quad (\text{B.1})$$

for  $t > s$ , where  $Y(t, s)$  is a given function, and where the initial value  $G(x, s, s)$  is also given.

The starting point of the method consists in writing the proportionality

$$dt = \frac{dx}{(x - 1)/\Lambda(t)} = \frac{dG}{(x - 1)(G - Y)}. \quad (\text{B.2})$$

The first equality in equation (B.2) yields

$$x - 1 = ye^{\tau(t)} \quad (\text{B.3})$$

with

$$\tau(t) = \int_0^t \frac{du}{\Lambda(u)} \quad (\text{B.4})$$

and where the constant  $y$  parametrizes the characteristic curves.

The second equality in equation (B.2) then reads

$$\frac{dG}{dt} = ye^{\tau(t)}(G - Y) \quad (\text{B.5})$$

which can be integrated by varying the constant. This yields, taking into account the initial condition,

$$G(x, t, s) = e^{(x-1)D(t,s)} G(1 + (x - 1)e^{\tau(s) - \tau(t)}, s, s) + \int_s^t du \mathcal{K}(x, t, u) Y(u, s) \quad (\text{B.6})$$

with the definitions

$$\begin{aligned} D(t, u) &= \int_u^t dv e^{\tau(v) - \tau(t)} \\ \mathcal{K}(x, t, u) &= (1 - x) e^{\tau(u) - \tau(t) + (x-1)D(t,u)}. \end{aligned} \quad (\text{B.7})$$

**Appendix C. Equilibrium thermodynamics**

Consider a system made of  $M$  boxes containing  $N$  particles. As in appendix A, we define a configuration  $\mathcal{C}$  of the system as given by the occupation numbers of the boxes:  $\mathcal{C} = \{N_1, N_2, \dots, N_M\}$ , with  $\sum_i N_i = N$ . The partition function of the system reads

$$Z_{M,N} = \frac{1}{N!} \sum_{\{N_i \geq 0, \sum_i N_i = N\}} \frac{N!}{\prod_i N_i!} e^{\beta \sum_i \delta_{N_i,0}}. \quad (\text{C.1})$$

The combinatorial weight inside the sum represents the number of microstates corresponding to each configuration  $\mathcal{C}$ , where a microstate is defined by assigning in which box lies each of the  $N$  particles. The overall  $1/N!$  in front of the sum takes account of the fact that a global relabelling of the identical particles does not affect equilibrium properties. This statistic, used, for example, for ideal gases of classical particles, is usually referred to as the Maxwell–Boltzmann statistic [16].

Inserting into equation (C.1) the contour integral representation  $\oint \frac{dz}{2\pi iz} z^{\sum_i N_i - N}$  for the constraint, we obtain

$$Z_{M,N} = \oint \frac{dz}{2\pi iz} s(z)^M z^{-N} \quad (\text{C.2})$$

with

$$s(z) = \sum_{k \geq 0} \frac{e^{\beta \delta_{k,0}}}{k!} z^k = e^z + e^\beta - 1. \quad (\text{C.3})$$

We have in particular  $Z_{M,N} = M^N/N!$  at infinite temperature.

In the thermodynamic limit ( $M, N \rightarrow \infty$ ), the density  $\rho = N/M$  being fixed, the method of steepest descent can be applied to the integral (C.2). The saddle-point equation reads  $ds(z)/dz = \rho s(z)/z$ . The saddle-point value  $z_c$  of  $z$ , which is by definition the thermodynamical fugacity of the model, is thus related to temperature and density through

$$e^\beta = 1 + (z_c/\rho - 1)e^{z_c}. \quad (\text{C.4})$$

The free energy of the model is an extensive quantity, and its value per box reads

$$-\beta F = \ln s(z_c) - \rho \ln z_c = z_c + (1 - \rho) \ln z_c - \ln \rho. \quad (\text{C.5})$$

Physical quantities can be derived along the same lines. For instance, by restricting the sum in equation (C.1) to the configurations  $\{N_i\}$  such that  $\sum_i N_i = N$  and  $N_1 = k$ , we obtain for a finite system

$$(f_k)_{M,N} = \frac{1}{Z_{M,N}} \frac{e^{\beta \delta_{k,0}}}{k!} \oint \frac{dz}{2\pi iz} s(z)^{M-1} z^{k-N} \quad (\text{C.6})$$

and in the thermodynamic limit

$$(f_k)_{\text{eq}} = \rho \exp(-z_c + \beta \delta_{k,0}) \frac{z_c^{k-1}}{k!}. \quad (\text{C.7})$$

We now restrict ourselves to the case  $\rho = 1$ , considered in the body of the paper. Equations (5.2) and (C.4) show that the fugacity  $z_c$  is identical to the equilibrium value  $\Lambda_{\text{eq}}$  of  $\Lambda(t)$ , while equation (5.1) for the occupation probabilities coincides with equation (C.7).

The low-temperature behaviour of various thermodynamical quantities, such as the fugacity  $\Lambda_{\text{eq}} = z_c$ , the internal energy  $E_{\text{eq}} = -(f_0)_{\text{eq}}$ , and the specific heat  $C_{\text{eq}} = -\beta^2 dE_{\text{eq}}/d\beta$ , is obtained by expanding the above results for  $\beta \rightarrow \infty$ :

$$\begin{aligned}\Lambda_{\text{eq}} &= \beta - \ln \beta + \frac{\ln \beta + 1}{\beta} + \frac{\ln^2 \beta - 1}{2\beta^2} + \dots \\ E_{\text{eq}} &= -1 + \frac{1}{\beta} + \frac{\ln \beta}{\beta^2} + \frac{\ln^2 \beta - \ln \beta - 1}{\beta^3} + \dots \\ C_{\text{eq}} &= 1 + \frac{2 \ln \beta - 1}{\beta} + \frac{3 \ln^2 \beta - 5 \ln \beta - 2}{\beta^2} + \dots\end{aligned}\tag{C.8}$$

The constant equilibrium specific heat  $C_{\text{eq}} \approx 1$  at low temperature is a somewhat pathological feature of the model [16], that is shared by several classical systems, such as the ideal gas and the harmonic oscillator.

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